2.2a nonlinear difference equations

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Def 2.1 For the 1st difference equation / 1st-order system

$x_{t+1} = f(x_t)$, $\quad X(t+1) = F(X(t))$

an equilibrium solution or steady-state solution is a constant solution $\bar{x}$ to the difference equation. i.e. $(\bar{X})$

$\bar{x} = f(\bar{x})$ / $\bar{X} = F(\bar{X})$

$\bar{x}$ and $\bar{X}$ are fixed pts of respectively f or F.

Notation: Let $f^t(x_0) = \underbrace{f \circ f \circ \cdots \circ f}_{t \text{ times}}(x_0)$. So, if $x_{t+1} = f(x_t)$, then $f^t(x_0) = x_t$

Def. 2.2 A periodic solution of period $m > 1$ of a difference eq $x_{t+1} = f(x_t)$ is a real-valued sol $\bar{x}_k$ satisfying

$f^m(\bar{x}_k) = \bar{x}_k$ and $f^i(\bar{x}_k) \neq \bar{x}_k$ for $i = 1, \dots, m-1$

An m-cycle is a set of pts $\{\bar{x}_1, \dots, \bar{x}_m\}$ where $f(\bar{x}_k) = \bar{x}_{k+1}$ and each pt $\bar{x}_k$ for $k = 1, \dots, m$ is a periodic solution of period m.

The set $\{\bar{x}_1, f(\bar{x}_1), \dots, f^{m-1}(\bar{x}_1)\}$ is the periodic orbit of $\bar{x}_1$.

Similar definitions for a first-order system $X(t+1) = F(X(t))$

Aside: If $\bar{x}_k$ is a periodic solution to $x_{t+1} = f(x_t)$ of period m, then $\bar{x}_k$ is a fixed pt of $f^m, f^{2m}, f^{3m}, \dots$

Aside: By def., a solution of period m can't have period $k < m$.

Def. 2.3a An equilibrium solution $\bar{x}$ of $x_{t+1} = f(x_t)$ is locally stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x_0 - \bar{x}| < \delta$, then if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x_0 - \bar{x}| < \delta$, then

$$|x_t - \bar{x}| = |f^t(x_0) - \bar{x}| < \varepsilon \qquad \forall t \geq 0.$$

If $\bar{x}$ is not stable, then it is unstable.

Def. 2.3b An equilibrium solution $\bar{x}$ of $x_{t+1} = f(x_t)$ is locally attracting if $\exists \gamma > 0$ s.t. for all x_0 s.t. $|x_0 - \bar{x}| < \gamma$,

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} f^t(x_0) = \bar{x}$$

Def. 2.3c The equilibrium solution $\hat{x}$ is locally asymptotically stable if it is both locally attracting and locally stable.

Important: It is possible to be locally attracting but not locally stable.

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